

# Floating Entanglement Witness Measure and Genetic Algorithm

A. Baghbanpourasl<sup>a</sup> <sup>\*</sup>, G. Najarbashi <sup>b</sup> <sup>†</sup>, M. Seyedkazemi <sup>c</sup> <sup>‡</sup>

<sup>a</sup>Department of Physics, Amir Kabir University, P. O. Box 15875-4413, Tehran, Iran.

<sup>b</sup>Department of Physics, Mohaghegh Ardabili University, Ardabil 56199-11367, Iran.

<sup>c</sup>IAU-Ardabil Branch, Young Researchers Club, Ardabil, Iran

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<sup>\*</sup>E-mail address: baghban@galaan.com

<sup>†</sup>E-mail address: najarbashi@tabrizu.ac.ir

<sup>‡</sup>E-mail address: mohsen.seyedkazemi@gmail.com

**Abstract**

In this paper based on the notion of entanglement witness, a new measure of entanglement called floating entanglement witness measure is introduced which satisfies some of the usual properties of a good entanglement measure. By exploiting genetic algorithm, we introduce a classical algorithm that computes floating entanglement witness measure. This algorithm also provides a method for finding entanglement witness for a given entangled state.

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# 1 Introduction

Entanglement is one of the most interesting properties of quantum mechanics and the key resource of some quantum information and quantum computation processes, such as teleportation, dense coding and quantum key distribution [1, 2]. However characterizing and measuring the entanglement has tantalized physicists since the earliest days of quantum mechanics, and even today there is no general qualitative and quantitative theory of entanglement.

Among the known criterions for distinguishing between separable and entangled states [3, 4, 5, 6, 7, 8, 9], entanglement witnesses (EWs) have a special importance for detecting the presence of entanglement. Finding EW that leads to solving separability problem, is an interesting but computationally a demanding job as it has been shown that the separability problem lies in the class of NP-hard problems [10]. There has been a lot of efforts for finding EW for given quantum state; some of them propose EW for some special cases [11] while others try to find approximate solutions by using different methods like semidefinite and linear programming [12, 13]. In addition to the question of "Is it entangled?" there is the question of "How much entangled?". Some measures of entanglement related to EW are introduced by Bertlemann *et al* [14], Brandao and Vianna [15, 16].

The aim of this paper is two fold: firstly by defining a slightly different definition than of EW which is called floating entanglement witness (FEW), we introduce a new measure of entanglement namely FEW measure. Secondly based on the genetic algorithm (GA) we offer an algorithm which computes FEW measure and also finds EW for every entangle state. GA is a powerful and intelligent technique for global optimization, adaptation and search problems [17, 18, 19]. It reveals its power especially when we are dealing with highly nonlinear and large search spaces. This technique is inspired by natural evolution of species which is based on *selection, inheritance* and *mutation*. Readers are referred to appendix I for an overview of GA.

The paper is organized as follows: In section 2 we briefly recall the definition of the EW and define our concept of the FEW measure. In section 3 an algorithm is proposed for finding FEW measure and EW for a given entangled state. Section 4 is devoted to some examples such as: Bell, Werner, mixture of GHZ and W and one parameter two-qutrit states. The paper is ended with a brief conclusion and two appendices.

## 2 FEW measure

As mentioned in the introduction one of the methods for detecting entanglement is applying EWs. Let us first recall the definition of entanglement, separability and EWs. A density matrix  $\rho$  is called *separable* or *unentangled* if there are positive  $p_i$ 's with  $\sum_i p_i = 1$  and product states  $|\psi_i\rangle = |\alpha_i^{(1)}\rangle|\alpha_i^{(2)}\rangle\dots|\alpha_i^{(n)}\rangle$  such that

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad (2.1)$$

otherwise it is called *entangled*.

**Definition 1.** A Hermitian operator  $\mathcal{W} \in \mathcal{B}(\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \dots \otimes \mathcal{H}_{d_n})$  (the Hilbert space of bounded operators) is called an EW detecting the entangled state  $\rho_e$  if  $\text{Tr}(\mathcal{W}\rho_e) < 0$  and  $\text{Tr}(\mathcal{W}\rho_s) \geq 0$  for all separable state  $\rho_s \in \mathcal{S}$ .

Therefore, if for state  $\rho$  we measure  $\text{Tr}(\mathcal{W}\rho) < 0$ , we can be sure that  $\rho$  is entangled.

This definition has a clear geometrical meaning. Thus, the set of states for which  $\text{Tr}(\mathcal{W}\rho) = 0$ , is a hyperplane in the set of all states, that cuts this set into two parts. In the part in which  $\text{Tr}(\mathcal{W}\rho) > 0$ , lies the set of all separable states. The other part (with  $\text{Tr}(\mathcal{W}\rho) < 0$ ) is the set of entangled states detectable by  $\mathcal{W}$ . From this geometrical interpretation it follows that for each entangled state  $\rho_e$ , there exist an entanglement witness detecting it. This statement is proved in [4].

**Definition 2.** A traceless Hermitian operator  $\mathcal{Z}$  is called a floating entanglement witness (FEW) for detecting the entangled state  $\rho_e$  if  $\text{Tr}(\mathcal{Z}\rho_e) < \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s)$  where  $\mathcal{S}$  is the set of separable states.

It is necessary to note that the existence of  $\mathcal{Z}$  comes from the existence of EW for every  $\rho_e$ . Due to the convexity of the separable states, the minimum in the above definition comes from the border of separable states (pure product states). One can easily show that

$$\mathcal{W} = \mathcal{Z} - \mu I \quad \text{where} \quad \mu = \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s), \quad (2.2)$$

is an EW since it satisfies both conditions of EW, i.e.:  $\text{Tr}(\mathcal{W}\rho_s) = \text{Tr}(\mathcal{Z}\rho_s) - \mu \geq 0$  for all separable  $\rho_s$  and  $\text{Tr}(\mathcal{W}\rho_e) = \text{Tr}(\mathcal{Z}\rho_e) - \mu < \text{Tr}(\mathcal{Z}\rho_s) - \mu \leq 0$ , which means that  $\text{Tr}(\mathcal{W}\rho_e) < 0$ .

A closely related problem to the EWs is characterization or quantification of entanglement by EWs (see [16]). Based on the concept of FEW we introduce a new computable FEW measure for quantifying entanglement of a given quantum state:

$$E(\rho) := \max \left\{ 0, \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}\rho) \right] \right\}, \quad (2.3)$$

where

$$\mathcal{A} := \{ \mathcal{Z} \in \mathcal{B}(\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \dots \otimes \mathcal{H}_{d_n}) \mid \text{Tr}(\mathcal{Z}) = 0, \|\mathcal{Z}\|_2 = 1 \},$$

and  $\|\mathcal{Z}\|_2 := \sqrt{\text{Tr}(\mathcal{Z}^\dagger \mathcal{Z})}$ , denotes the Hilbert-Schmidt norm. The FEW measure fulfills the following usual requirements of an entanglement measure [20]:

**Proposition 3.**  $E(\rho)$  satisfies the following properties:

(i) For every separable state  $\sigma_s \in \mathcal{S}$ ,  $E(\sigma_s) = 0$ .

(ii) Local unitary operations leave  $E(\rho)$  invariant, i.e,

$$E(\rho) = E(U_1^\dagger \otimes \dots \otimes U_n^\dagger \rho U_1 \otimes \dots \otimes U_n). \quad (2.4)$$

(iii) FEW measure dose not increase under local operations and classical communication (LOCC) protocols [21], i.e.,

$$E(\mathcal{E}(\rho)) \leq E(\rho). \quad (2.5)$$

(iv) The FEW measure is a convex function, i.e.,

$$E(\lambda\rho + (1-\lambda)\sigma) \leq \lambda E(\rho) + (1-\lambda)E(\sigma). \quad (2.6)$$

(v) FEW measure is a continuous function.

*Proof:* (i) It is easy to see that for every  $\mathcal{Z}$  we have  $\min_{\rho_s} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}\sigma_s) \leq 0$  hence  $E(\sigma_s) = 0$ .

(ii) (2.4) follows from the invariance of  $\mathcal{S}$  and  $\|\cdot\|_2$  under local unitary operations.

(iii) Although we did not find rigorous proof for monotonicity of FEW measure under general LOCC maps  $\mathcal{E}$ , it can be proved for isometry ones for which we require  $\mathcal{E}^\dagger \mathcal{E} = \mathcal{E} \mathcal{E}^\dagger = I$ . To see this we note that

$$\begin{aligned} E(\mathcal{E}(\rho)) &= \max \left\{ 0, \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}\mathcal{E}(\rho)) \right] \right\} \quad (\text{by definition}) \\ &= \max \left\{ 0, \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\mathcal{E}^\dagger(\rho_s) \in \mathcal{S}} \text{Tr}(\mathcal{E}^\dagger(\mathcal{Z})\mathcal{E}^\dagger(\rho_s)) - \text{Tr}(\mathcal{E}^\dagger(\mathcal{Z})\rho) \right] \right\} \quad (\text{isometry}) \\ &= \max \left\{ 0, \max_{\mathcal{Z}' \in \mathcal{E}^\dagger(\mathcal{A})} \left[ \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}'\rho_s) - \text{Tr}(\mathcal{Z}'\rho) \right] \right\} \quad (\mathcal{E}^\dagger(\mathcal{Z}) \rightarrow \mathcal{Z}') \\ &\leq \max \left\{ 0, \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}\rho) \right] \right\} = E(\rho) \quad (\mathcal{E}^\dagger(\mathcal{A}) \subseteq \mathcal{A}) \end{aligned}$$

where in the second equality we use the fact that  $\mathcal{E}^\dagger(\mathcal{S}) = \mathcal{S}$ , [22] and  $\text{Tr}(\mathcal{Z}\mathcal{E}(\rho)) = \text{Tr}(\mathcal{E}^\dagger(\mathcal{Z})\rho)$ .

(iv) To prove the convexity we note that

$$\begin{aligned} E(\lambda\rho + (1-\lambda)\sigma) &= \max \left\{ 0, \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}(\lambda\rho + (1-\lambda)\sigma)) \right] \right\} \\ &= \max \left\{ 0, \max_{\mathcal{Z} \in \mathcal{A}} \left[ \lambda \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) + (1-\lambda) \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) - \lambda \text{Tr}(\mathcal{Z}\rho) - (1-\lambda) \text{Tr}(\mathcal{Z}\sigma) \right] \right\} \end{aligned}$$

$$\begin{aligned} &\leq \lambda \max \left\{ 0, \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}\rho) \right] \right\} + (1-\lambda) \max \left\{ 0, \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}\sigma) \right] \right\} \\ &\quad = \lambda E(\rho) + (1-\lambda) E(\sigma). \end{aligned}$$

where the inequality comes from the fact that,  $\max_{\Omega} (f_1 + f_2) \leq \max_{\Omega} f_1 + \max_{\Omega} f_2$ , for every function  $f_1, f_2$  defined on a given set  $\Omega$ .

(v) To prove continuity, for entangled states  $\rho$  and  $\sigma$  we suppose that  $E(\rho) > E(\sigma)$ , without loss of generality. By definition

$$E(\rho) = \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\rho_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}\rho) \right] = \text{Tr}(\mathcal{Z}_\rho \rho_{s_0}) - \text{Tr}(\mathcal{Z}_\rho \rho),$$

where  $\mathcal{Z}_\rho$  and  $\rho_{s_0}$  are the  $\mathcal{Z} \in \mathcal{A}$  and  $\rho_s \in \mathcal{S}$  for which the maximum occurs in the above equation. On the other hand we have

$$\begin{aligned} E(\sigma) &= \max_{\mathcal{Z} \in \mathcal{A}} \left[ \min_{\sigma_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\sigma_s) - \text{Tr}(\mathcal{Z}\sigma) \right] \geq \forall \mathcal{Z} \left[ \min_{\sigma_s \in \mathcal{S}} \text{Tr}(\mathcal{Z}\sigma_s) - \text{Tr}(\mathcal{Z}\sigma) \right] \\ &\implies E(\sigma) \geq \text{Tr}(\mathcal{Z}_\rho \rho_{s_0}) - \text{Tr}(\mathcal{Z}_\rho \sigma) \end{aligned}$$

Therefore

$$E(\rho) - E(\sigma) \leq -\text{Tr}(\mathcal{Z}_\rho \rho) + \text{Tr}(\mathcal{Z}_\rho \sigma) = \text{Tr}(\mathcal{Z}_\rho (\sigma - \rho)) \leq \|\mathcal{Z}_\rho\|_2 \|\sigma - \rho\|_2 = \|\sigma - \rho\|_2 = \epsilon$$

for some real number  $\epsilon \geq 0$ . In the last inequality we have used Cauchy-Schwartz inequality for Hilbert-Schmidt distance.

Another usual property of every measure of entanglement is the *additivity*  $E(\rho^{\otimes n}) = nE(\rho)$  or *subadditivity*  $E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma)$  problem which for FEW measure remains open for debate.

### 3 Finding FEW measure and EWs using GA

This section deals with application of GA in finding FEW measure and EWs. At the first stage the problem is finding a FEW for a given  $\rho$ . The fitness function is defined as

$$\mathcal{F}(\rho, \mathcal{Z}) := \min_{\rho_s} \text{Tr}(\mathcal{Z}\rho_s) - \text{Tr}(\mathcal{Z}\rho). \quad (3.7)$$

The task is to maximize  $\mathcal{F}$  over  $\mathcal{Z}$  by using the GA, which implies that GA tends to make  $Tr(\mathcal{Z}\rho)$  smaller than  $\min Tr(\mathcal{Z}\rho_s)$ , as well as it increases the difference between these two terms in each chromosome. As it is clear from the fitness function we need a second optimization procedure that finds  $\min Tr(\mathcal{Z}\rho_s)$ , for each chromosome. In this subprogram, quasi-Newton (QN) optimization method is used by employing optimization function UMINF of IMSL math library. For this reason we calculate  $Tr(\rho_s \mathcal{Z})$  for a large number,  $N_{QN1}$ , of random  $\rho_s$ 's then use  $N_{QN2}$  of them which have smaller  $Tr(\rho_s \mathcal{Z})$  as initial points for running QN. The smallest resulting value is chosen as  $\min_{\rho_s} Tr(\mathcal{Z}\rho_s)$ .

Regarding the above considerations the algorithm goes as follows:

1. Read input density matrix  $\rho$ .
2. Populate an initial pool of random  $\mathcal{Z}$ 's (chromosomes), i.e. produce  $\mathcal{Z}$ 's with random parameters.
3. (a) Find  $f_1 := \min_{\rho_s} Tr(\rho_s \mathcal{Z})$  for each  $\mathcal{Z}$  by using QN.  
 (b) Find  $f_2 := Tr(\rho \mathcal{Z})$  for every  $\mathcal{Z}$  in the pool.  
 (c) Compute the fitness function  $\mathcal{F} = f_1 - f_2$ , for every  $\mathcal{Z}$ .
4. Produce the new generation by doing selection, crossover and mutation.
5. Go to step (3) until the stop criteria is met, i.e., maximum number of iterations is run.
6. (a) Select the chromosome ( $\mathcal{Z}$ ), with maximum  $\mathcal{F}$   
 (b) If  $\mathcal{F} > 0$  then  $\mathcal{F}$  is FEW measure and  $\mathcal{Z}$  is FEW.  
 (c) Compute  $\mathcal{W} = \mathcal{Z} - \mu I$  (with  $\mu = f_1$ ) as an EW for detecting  $\rho$ .
7. if  $\mathcal{F} \leq 0$ ,  $\rho$  is separable.

It is useful to mention that even if we just want to find EW, it is better to find it through FEW. An EW,  $\mathcal{W}$ , must satisfy the conditions in definition 1, i.e. a  $\mathcal{W}$  which  $Tr(\mathcal{W}\rho) < 0$

for the given  $\rho$ , subjected to the constraint  $Tr(\mathcal{W}\rho_s) \geq 0$  for all separable states  $\rho_s$ 's. The latter condition puts our problem in the field of constrained problems. GA is adaptable to the constrained problems by defining a proper fitness function. But the constraint of the definition 1 slows down the algorithm and therefore instead of it we use definition 2 which breaks the constraint and greatly speeds up the algorithm.

The above algorithm can find a FEW, an EW and FEW measure for any arbitrary entangled density matrix. The robustness of the algorithm will be clear in the next section with various examples.

## 4 Some examples

In this section we discuss some interesting cases which can help to clarify the subject. In all cases we use the fact that any traceless Hermitian operator acting on the Hilbert space  $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \dots \otimes \mathcal{H}_{d_n}$  can be expressed by identity operator  $I_{d_1 d_2 \dots d_n}$  and generators of Lie algebra  $su(d_i)$ ,  $\lambda_j^{(d_i)}$ , as

$$\mathcal{Z} = \sum_{i_1=0}^{d_1^2-1} \sum_{i_2=0}^{d_2^2-1} \dots \sum_{i_n=0}^{d_n^2-1} \tau_{i_1 i_2 \dots i_n} \lambda_{i_1}^{(d_1)} \otimes \lambda_{i_2}^{(d_2)} \otimes \dots \otimes \lambda_{i_n}^{(d_n)}, \quad \tau_{00\dots 0} = 0, \quad (4.8)$$

where  $\lambda_0^{(d_i)} = I_{d_i}$  and  $\tau_{i_1 i_2 \dots i_n} \in \mathbb{R}$ .

Typically in GAs, there is no strict rule for choosing parameters of them. Probability of crossover  $P_c$  is usually chosen 0.7 and probability of mutation  $P_m$ , is a very small number in  $10^{-3}$  order. Number of population  $N_{GA}$  is usually several hundreds to several thousands.

In the following examples, crossover is two-point,  $P_c$  is 0.7,  $P_m$  is 0.007 and  $N_{GA}$  is chosen 20 (for 2 qubit states) and 10 (for other states) times bigger than the number of parameters of the problem. Convergence of GA in our problem starts, depending on the quantum state, from 20 iteration to higher ones. Therefore the maximum number of generations  $G_{GA}$ , in most examples is chosen 300 which is big enough for finding proper solutions. Also, there are no

lower and upper limits for parameters of QN except that choosing very small numbers make QN less successful in finding global minimum and very big numbers make the algorithm very slow. In this program,  $N_{QN1}$  is chosen about 100 times bigger than the number of parameters of the general form of pure separable states and  $N_{QN2}$  is chosen about this number of parameters.

## 4.1 Two-qubit systems

It is important both theoretically and experimentally to study entanglement of qubit systems and to provide EWs to verify that in a given state, entanglement is really present.

Here we find an EW for a given two-qubit density matrix. The most general form of a traceless Hermitian operator in the space of two-qubit states can be written as:

$$\mathcal{Z} = \sum_{i,j=0}^3 \tau_{ij} \sigma_i \otimes \sigma_j, \quad \tau_{00} = 0, \quad (4.9)$$

where  $\tau_{ij} \in \mathbb{R}$ ,  $\sigma_0 = I_2$  and  $\sigma_i$ 's with  $i = 1, 2, 3$ , are usual Pauli matrices. The equation (4.9) has 15 parameters ( $\tau_{ij}$ 's) which have to be determined by GA. The range of  $\tau_{ij}$ 's is determined by the condition  $\|\mathcal{Z}\|_2 = 1$  in every example. Each of the  $\tau_{ij}$ 's is encoded in a 15 bit binary number. Therefore every  $\mathcal{Z}$  can be encoded in a chromosome of 225 bits. Parameters of each chromosome are passed to the QN. In the QN, for finding  $f_1$ , it is sufficient to take this minimum over the  $\rho_s$ 's in the form  $\rho_s = |\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2|$  which can be parameterized as:

$$|\phi_j\rangle = \cos(\beta_j)|0\rangle + e^{i\alpha_j} \sin(\beta_j)|1\rangle, \quad j = 1, 2 \quad (4.10)$$

where  $\alpha_j \in [0, 2\pi]$  and  $\beta_j \in [0, \frac{\pi}{2}]$ . QN algorithm minimizes  $Tr(\mathcal{Z}\rho_s)$  by determining these four parameters.

### 4.1.1 Bell states

We begin with the simplest case that is finding an EW for pure Bell state  $|\psi_{00}\rangle\langle\psi_{00}|$ , where  $|\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ . Crossover is two-point, probability of crossover ( $P_c$ ) is 0.7, probability

of mutation ( $P_m$ ) is 0.007 and numbers of population and generations for GA are  $N_{GA} = 350$  and  $G_{GA} = 80$  respectively and parameters of QN are  $N_{QN1} = 400$  and  $N_{QN2} = 5$ . The program gives us the following solution:

$$\mathcal{W} = \begin{pmatrix} 0 & 0.296 & 0.280 & -0.289 + 0.001i \\ 0.296 & 0.575 & 0.288 - 0.001i & 0.283 \\ 0.280 & 0.288 + 0.001i & 0.578 & 0.292 \\ -0.289 - 0.001i & 0.283 & 0.292 & 0 \end{pmatrix}$$

with  $E(\rho) = 0.577$ . If we don't impose the condition  $\|\mathcal{Z}\|_2 = 1$ , we have:

$$\mathcal{W} = \begin{pmatrix} 0 & 0.003i & 0.002i & -1 + 0.014i \\ -0.003i & 0.999 & 0 & 0.002i \\ -0.002i & 0 & 0.999 & 0 \\ -1 - 0.014i & -0.002i & 0 & 0 \end{pmatrix}$$

This is very similar to the EW corresponding to *reduction map* [23]:

$$\mathcal{W}_{red} = I - 2|\psi_{00}\rangle\langle\psi_{00}| = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad (4.11)$$

Of course choosing any Bell state

$$|\psi_{ij}\rangle := \sigma_z^i \otimes \sigma_x^j |\psi_{00}\rangle, \quad i, j = 0, 1, \quad (4.12)$$

the GA yields an EW similar to reduction EW,  $\mathcal{W}_{red} = I - 2|\psi_{ij}\rangle\langle\psi_{ij}|$ .

#### 4.1.2 Werner states

One of the most important degraded Bell states is Werner state [24]. A Werner state in  $2 \otimes 2$  system takes the following form:

$$\rho_W = F|\psi_{00}\rangle\langle\psi_{00}| + \frac{1-F}{3}(|\psi_{10}\rangle\langle\psi_{10}| + |\psi_{01}\rangle\langle\psi_{01}| + |\psi_{11}\rangle\langle\psi_{11}|), \quad 0 \leq F \leq 1 \quad (4.13)$$

where  $|\psi_{ij}\rangle$ 's are Bell states defined in Eq. (4.12). The Werner state  $\rho_W$  is characterized by a single real parameter  $F$  called fidelity. This quantity measures the overlap of Werner state with a Bell state  $|\psi_{00}\rangle$ . The  $\rho_W$  is separable for  $0 \leq F \leq \frac{1}{2}$  and is entangled for  $\frac{1}{2} < F \leq 1$ . In these cases which the EW is complicated to write down, we give only the FEW measure (see Fig. 1).

## 4.2 Three-qubit systems

The procedure for three-qubit states is similar to the one described in the previous subsection. It is clear that every three-qubit traceless Hermitian operator  $\mathcal{Z}$  can be written by tensor product of Pauli operators as

$$\mathcal{Z} = \sum_{i,j,k=0}^4 \tau_{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k, \quad \tau_{000} = 0 \quad (4.14)$$

where the number of parameters  $\tau_{ijk} \in \mathbb{R}$ , is 63, so every  $\mathcal{Z}$  can be encoded in a string of  $15 \times 63 = 945$  bits. Similar to two-qubit case, we work with pure separable states  $\rho_s = |\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2| \otimes |\phi_3\rangle\langle\phi_3|$ , for finding  $\min_{\rho_s} \text{Tr}(\mathcal{Z}\rho_s)$ , where  $|\phi_j\rangle$  is defined the same as Eq. (4.10). Therefore we have 6 parameters for QN.

For example consider the mixture of  $|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$  and  $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ :

$$\rho_q = q|GHZ\rangle\langle GHZ| + (1 - q)|W\rangle\langle W|, \quad 0 \leq q \leq 1 \quad (4.15)$$

Results for FEW measure which are shown in Fig. 2 are in agreement with *witnessed entanglement* [16].

## 4.3 One parameter two-qutrit state

The description of qutrit systems is very similar to the one for qubits. For two-qutrit states the operator  $\mathcal{Z}$  can be expressed as  $\mathcal{Z} = \sum_{i,j=0}^8 \tau_{ij} \lambda_i \otimes \lambda_j$ ,  $\tau_{00} = 0$ , where  $\lambda_i$ 's with  $i = 1, \dots, 8$

are the well-known Gell-Mann matrices (see appendix II). The number of parameters  $\tau_{ij} \in \mathbb{R}$ , is 80, so every  $\mathcal{Z}$  could be encoded in a string of  $15 \times 80 = 1200$  bits. We work with pure separable states  $\rho_s = |\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2|$ , for finding  $\min Tr(\mathcal{Z}\rho_s)$ , where

$$|\varphi_j\rangle = e^{i\eta_j} \sin(\theta_j) \sin(\phi_j) |0\rangle + e^{i\xi_j} \sin(\theta_j) \cos(\phi_j) |1\rangle + \cos(\theta_j) |2\rangle, \quad j = 1, 2$$

and  $\eta_j, \xi_j \in [0, 2\pi]$  and  $\theta_j, \phi_j \in [0, \frac{\pi}{2}]$ . As last example consider the one parameter two-qutrit density matrix

$$\rho_\alpha = \frac{2}{7}|\phi_+\rangle\langle\phi_+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-, \quad 2 \leq \alpha \leq 5 \quad (4.16)$$

where

$$\begin{aligned} |\phi_+\rangle &= \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \\ \sigma_+ &= \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|) \\ \sigma_- &= \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|) \end{aligned} \quad (4.17)$$

It has been shown that  $\rho_\alpha$  is separable if  $2 \leq \alpha \leq 3$  and entangled if  $3 < \alpha \leq 5$ , [6]. FEW measure for these states is shown in Fig. 3.

## 5 Conclusion

We have introduced a new measure of entanglement called FEW measure, to quantify entanglement of quantum states which is based on a slightly different definition than of EW. Some properties of every measure of entanglement were proved for this measure. Using GA we proposed an algorithm to compute FEW measure for multipartite systems. For several examples including two-qubit, three-qubit and two-qutrit systems, the results of this method were illustrated. As in the examples, this method is capable of finding numerically these operators and quantity for every quantum state with any number of dimensions and parties. This algorithm also provided a method for finding EW for any entangled state.

## Appendix I

*GA:* GA is a search algorithm, based on natural selection and genetics [17]. In every GA there is a population of individuals, named chromosomes which a possible solution is encoded in each of them. A number is assigned to every chromosome showing its fitness to survive and reproduce. This number is calculated by fitness function. To form a new generation, chromosomes are selected based on their fitness, i.e., the fitter chromosome has more chance to be selected for reproduction. Tournament selection is one of the selection methods which selects the fittest chromosome in a number of randomly selected chromosomes. Selected chromosomes (parents) are subjected to genetic operations of crossover and mutation to reproduce offsprings. Crossover is combination of two parents with a probability,  $P_c$ , creating one or two new chromosomes. Parents interchange some parts of their chromosomes at some randomly chosen places, e.g., in two-point crossover, two places are chosen randomly then everything between these places is swapped between parents. Then offsprings are passed to mutation stage which is a genetic operation that alters some places of each chromosome at a very low probability of occurrence denoted by  $P_m$ . Offsprings create a new generation and this procedure continues until stop condition has been reached, e.g., a solution is found or the user specified maximum number of generations is evolved.

## Appendix II

For the reader convenience we present here explicit realization of generators  $\lambda_1, \dots, \lambda_8$  of Lie algebra  $su(3)$ :

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

## References

- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.
- [2] The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation and Quantum Computation, edited by D. Bouwmeester, A. Ekert, and A. Zeilinger, Springer, New York, 2000.
- [3] A. Peres, Separability criterion for density matrices, *Phys. Rev. Lett.* 77 (1996) 1413.
- [4] M. Horodecki, P. Horodecki, and R. Horodecki, Separability criterion and local information in separable states, *Phys. Lett. A* 223 (1996) 1.
- [5] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, *Phys. Rev. Lett.* 80 (1998) 2245.
- [6] P. Horodecki, M. Horodecki, and R. Horodecki, Bound entanglement can be activated, *Phys. Rev. Lett.* 82 (1999) 1056.
- [7] B. Kraus, J. I. Cirac, S. Karnas, M. Lewenstein, Separability in  $2 \times N$  composite quantum systems, *Phys. Rev. A* 61 (2000) 062302.

- [8] M. A. Nielsen and J. Kempe, Separable states are more disordered globally than locally, *Phys. Rev. Lett.* 86 (2001) 5184.
- [9] M. Lewenstein, B. Kraus, P. Horodecki, J. I. Cirac, Characterization of separable states and entanglement witnesses, *Phys. Rev. A* 63 (2001) 044304.
- [10] L. Gurvits, Classical deterministic complexity of Edmonds' Problem and quantum entanglement, in *Proceedings of the 35th ACM Symposium on the Theory of Computing*, ACM Press, New York, 2003, pp. 10-19.
- [11] G. Tóth, and O. Gühne, Entanglement detection in the stabilizer formalism, *Phys. Rev. A* 72 (2005) 022340.
- [12] R. O. Vianna and A. C. Doherty, Distillability of Werner states using entanglement witnesses and robust semidefinite programs, *Phys. Rev. A* 74 (2006) 052306.
- [13] M. A. Jafarizadeh, G. Najarbashi, H. Habibian, Manipulating multiqudit entanglement witnesses by using linear programming, *Phys. Rev. A* 75 (2007) 052326.
- [14] R. A. Bertlmann, H. Narnhofer and W. Thirring, Geometric picture of entanglement and Bell inequalities, *Phys. Rev. A* 66 (2002) 032319.
- [15] F. G. S. L. Brandao and R. O. Vianna, Witnessed entanglement, *IJQI*, 4 (2006) 331.
- [16] F. G. S. L. Brandao, Quantifying entanglement with witness operators, *Phys. Rev. A* 72 (2005) 022310.
- [17] M. Mitchell, *An Introduction to Genetic Algorithms*, A Bradford Book, The MIT Press, Cambridge, MA, 1999.
- [18] R.V. Ramos, R.F. Souza, Calculation of the quantum entanglement measure of bipartite states, based on relative entropy, using genetics algorithms, *J. Comput. Phys.* 175 (2002) 576.

- [19] R. V. Ramos, Numerical algorithms for use in quantum information, *Journal of Computational Physics* 192 (2003) 95.
- [20] V. Vedral and M. B. Plenio, Entanglement measures and purification procedures, *Phys. Rev. A* 57 (1998) 1619.
- [21] N. Gisin, Hidden quantum nonlocality revealed by local filters, *Phys. Lett. A* 210 (1996) 151.
- [22] Assuming the isometry of  $\mathcal{E}$ , it is easy to show that  $\mathcal{E}$  and subsequently  $\mathcal{E}^\dagger$  are one to one and unto maps. By applying  $\mathcal{E}^\dagger$  on the both sides of the equality  $\mathcal{E}(\rho_s) = \mathcal{E}(\sigma_s)$  and using the isometry condition we get  $\rho_s = \sigma_s$  which implies  $\mathcal{E}$  is one to one. On the other hand for every  $\rho_s \in \mathcal{S}$  there exists  $\sigma_s \in \mathcal{S}$  such that  $\mathcal{E}(\sigma_s) = \rho_s$ , since it is sufficient to take  $\sigma_s = \mathcal{E}^\dagger(\rho_s) \in \mathcal{S}$  which in turn implies that  $\mathcal{E}$  is an unto map. Hence  $\mathcal{E}^\dagger(\mathcal{S}) = \mathcal{E}(\mathcal{S}) = \mathcal{S}$ .
- [23] W. Hall, Multipartite reduction criteria for separability, *Phys. Rev. A* 72 (2005) 022311.
- [24] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, *Phys. Rev. A* 40 (1989) 4277.

### Figure Captions

**Fig1:** Caption:  $E(\rho)$  versus fidelity  $F$  for Werner state  $\rho_w$ . Parameters of program:  $P_m = 0.007$ ,  $P_c = 0.7$ ,  $N_{GA} = 320$ ,  $G_{GA} = 300$ ,  $N_{QN1} = 400$  and  $N_{QN2} = 5$ . Resolution time for each generation is 1 seconds on a Celeron 2 GHz PC.

**Fig2:** Caption:  $E(\rho)$  versus  $q$  for GHZ-W mixture state  $\rho_q$ . Parameters of program:  $P_m = 0.007$ ,  $P_c = 0.7$ ,  $N_{GA} = 640$ ,  $G_{GA} = 300$ ,  $N_{QN1} = 500$  and  $N_{QN2} = 8$ . Resolution time for each generation is 3.75 seconds on a Celeron 2 GHz PC.

**Fig3:** Caption:  $E(\rho)$  versus  $\alpha$  for two-qutrit state  $\rho_\alpha$ . Parameters of program:  $P_m = 0.007$ ,  $P_c = 0.7$ ,  $N_{GA} = 810$ ,  $G_{GA} = 300$ ,  $N_{QN1} = 800$  and  $N_{QN2} = 10$ . Resolution time for each generation is 45 seconds on a Celeron 2 GHz PC.





